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FROM ARCHIMEDES TO POWERATIVE INTEGRAL

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Abstract. The paper is devoted to one direction of modern mathematics taking its origin from additive analysis, later from multiplicative analysis and, finally, powerative analysis. For each case there is given a direct operation, an integral and the inverse operative.

Keywords: additive derivative, additive integral, multiplicative derivative, multiplicative integral, powerative integral, powerative derivative.

AMS Subject Classification: 34E05.

1. Introduction.

As known, the additive integral is known for a very long time., even 2 thousand years ago. If Archimedes wanted to define the volume of some element (subject) then he split this element into small pieces and summed them, i.e. he found a Darboux sum to find a volume [11]. What concerns the additive derivative then it appeared in Newton and Leibnitz's time [15], [18]. These operations are related with additive analysis and have the additive property, the integral of a sum is a sum of integrals and the derivative of a sum is a sum of derivatives; a constant can be taken under the sign of integral as well as it can be taken out of the sign of the integral. The multiplicative analysis has appeared comparatively recently; the multiplicative derivative, multiplicative integral and their simplest properties are given in Gantmacher's book [12]. Here the multiplicative property holds true, i.e. an integral of a product is a product of integrals and the derivative of a product is the product of derivatives.

At last, the third one – the powerative analysis - belongs to us. We have defined the powerative integral and the powerative derivative.

Every science must be based on the axiomatic. In this sense the first science built axiomatically is mathematics. Nowadays the quant mechanics which is over 75% mathematics is being axiomatized.

2. The statement of the problem.

We start from arithmetic. The main elements of arithmetic are natural numbers N={1,2,3, …} and operation of addition "+". Some historians (who are engaged with mathematics) consider that they are sent by the Lord, i.e. they are main concepts that have no definition. It is known that if to take an arbitrary pair of elements from N then their sum belongs to N, i.e. $\forall m,n \in N \Rightarrow m+n \in N$. Further, the summation has the additivity property (commutability): $\forall m,n \in N \Rightarrow m+n = n+m \in N$. Thus the direct operation "+" is defined on the first stage. Now, based on the direct operation of the first stage we introduce the operation inverse to addition – the operation of subtraction "-". For $\forall m,n \in N \quad m-n$ is defined as such number k that $k+n=m$ that by definition the first addend is $k+n=m$. Now there arises a question: if we are given $k+n=m$ then how to define the second addend "*n*". All the listeners answered that $n=m-k$ but it is wrong as by the definition we said that for determination of the first addend we must subtract the second addend from the sum. So, if we are given $k+n=m$ then based on the commutability property of the sum we introduce we introduce it as $n+k=m$ and then, based on the introduced definition, we define the first addend as $n=m-k$. For completing the first stage we consider the difference of an arbitrary pair of elements from N. i.e. we define $m-n$ for $\forall m,n \in N$. If $m>n$ then $m-n \in N$. If $m \le n$ then $m-n \notin N$. In this case $m-n$ is considered a number (either "0" or negative numbers) and adding these numbers to *N* we come to the extension of set N which is a set of whole numbers $Z=\{....-2,-1,0,1,2,...\}$. Thus, the first stage is completed and we have obtained the extension $Z \supset N$. This extension is obtained by means of the inverse operation of the first stage – subtraction "-". The condition of commutability of addition provides the uniqueness of the inverse operation of subtraction, i.e. if $k+n=m$ then the operation of subtraction is only necessary for the definition of k or n.

3. The second stage.

The direct operation of this stage is multiplication. This operation depends on us. We can define the multiplication arbitrarily. For instance, if $m<n$ and $m,n \in N$ then by definition

$$m \cdot n = m+(m+1)+(m+2)+...+(n-1)+n = \sum_{k=m}^{n} k . \tag{1}$$

Taking into account that the definition (1) for the product has the property of commutability, i.e.

$$m \cdot n = n \cdot m , \tag{2}$$

then the operation inverse to multiplication will be the unique one too and we will be able to continue this process. The author accepts the definition of multiplication known long ago, i.e. "the sum of the same numbers is product", e.g. $5 \cdot 3 = 5+5+5$ or $5 \cdot 3 = 3+3+3+3+3$. As known, the definition of multiplication has the property of commutability. Thus,

$$\forall m,n \in Z \;\Rightarrow\; m \cdot n \in Z , \tag{3}$$

$$\forall m,n \in Z \;\Rightarrow\; m \cdot n = n \cdot m . \tag{4}$$

So that the direct operation of multiplication doesn't give any extension.

Now, based on multiplication, we define the inverse operation of "division" as follows: for $\forall m \in Z, n \in N$, "m" devised by n is considered as such a number $k \in Z$ for which

$$k \cdot n = m. \tag{5}$$

Then from (5) by definition the first multiplier

$$k = m : n. \tag{6}$$

If we want to define the second multiplier n from (5) then it is necessary to apply property (4), i.e.

$$n \cdot k = m,$$

and to define by definition of the first multiplier as follows

$$n = m : k.$$

Thus, if to take $\forall m \in Z, n \in N$ then $m : n$ doesn't belong to N then we consider it as a number and add it to Z, i.e. we extend Z to the set of rational numbers Q.
So, similar to the above the inverse operation of division gives us extension of the set of whole numbers: extension $N \subset Z \subset Q$.

By this four arithmetic operations are completed.

4. The third stage.

The direct operation of the third stage is exponentiation . Similar to (1), we can define the exponentiation arbitrarily. But the author accepts the definition which was familiar before him. Exponentiation is product of equal numbers, i.e.

$$n^3 = n \cdot n \cdot n, \ n \in N \ \ or \ \ n \in Z.$$

Taking into account that $2^3 = 8$ and $3^2 = 9$, i.e. $2^3 \neq 3^2$ - the commutability is absent in the exponentiation then the inverse operation will not be unique (two inverse operations).

Now we shall introduce two operations inverse to the exponentiation. If

$$m^n = k, \tag{7}$$

then

$$m = \sqrt[n]{k}, \ n = \log_m k. \tag{8}$$

By this 7 algebraic operations are completed.

It is easy to show that the root of the equation

$$x \cdot x = x^2 = 2$$

doesn't belong to Q, i.e. if $x^2 = 2$ then $x = \sqrt{2} \in Q$. Thus, we obtain a new number and call it a real number and in the literature we come to the following extension:

$N \subset Z \subset Q \subset R$.

But we divide the passage from Q to R into the following stages [4]. We designate these numbers which are obtained by means of radicals as R_1, i.e. the elements of N and Z as *m*, *n* or *k*, the elements of Q are designated as $\frac{m}{n}$ where $m \in Z,\ n \in N$.

The elements of set R_1 are denoted as $\sqrt[k]{\frac{m}{n}},\ m \in Z,\ n \in N, k \in N$; if k is an even number then $m \geq 0$; if k is an odd number then $m \in Z$.

5. The fourth stage.

Let us define the direct operation of this stage as the exponentiation on the left, i.e.

$$ {}^{4}3 = 3^{3^{3^{3}}} . \tag{9}$$

Taking into account the noncommutability of this operation we'll have two inverse operations which we denote as $\sqrt{}$ and $\log a$.

In paper [4] it was proved that the solution of the equation

$$ {}^{2}x = x^{x} = 2 \tag{10}$$

doesn't belong to $\mathbf{R_1}$, i.e. the solution of (10) cannot be represented by means of ordinary roots in the form

$$\sqrt[k]{\frac{m}{n}} ,$$

where $m \in Z,\ n \in N,\ k \in N$. That's why we obtained the following extension

$$N \subset Z \subset Q \subset R_1 \subset R_2 .$$

Continuing this process until the infinity we come (by means of the limits) to the set of real numbers R.

By the way, we have marked that in [5] we got to a space of fractional dimension. It is quite different direction of the modern mathematics.

Thus, the started by us development of the number set has brought us to the following step-by-step development.

Now let us consider a pair of direct successive operations

$$\sum_{k=1}^{n} f(x_k)\Delta x_k . \tag{11}$$

This is a sum of products. As shown by an arrow in the scheme below, first we find products then the summation. It is easy to see, that (11) reminds of a Darboux sum from the analysis [1], i.e.

$$\lim_{|\Delta x_k|\to 0}\sum_{k=1}^{n} f(x_k)\Delta x_k = \int_a^b f(x) \ , \tag{12}$$

which is an additive continuous integral known from both a secondary school course and a higher school. Now we consider the inverse operation

$$\frac{f(x+h)-f(x)}{h} \ , \tag{13}$$

which is a ratio, i.e. at first the subtraction is performed and then the division (as is shown by an arrow). It is easy to see that this reminds us of a derivative, i.e.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) \ . \tag{14}$$

It is an additive continuous derivative known from both a secondary school course and a higher school as well.

6. Continuous Multiplicative Analysis.

Now let us go up by one step: consider two direct operations "product of powers":

$$\prod_{k=1}^{n} f(x_k)^{\Delta x_k} \ . \tag{15}$$

The multiplicative analysis has been defined recently. In [12] on pp.3-4 there is defined a multiplicative derivative, a multiplicative integral and their simplest properties.

Multiplicative integral in a continuous case is

$$\int_a^b f(x)^{dx} = \lim_{\substack{\max|\Delta x_k|\to 0 \\ n\to\infty}} \prod_{k=0}^{n} f(x_k)^{\Delta x_k} \tag{16}$$

and multiplicative derivative in continuous case is

$$f^{[I]}(x) = \lim_{h\to 0}\sqrt[h]{\frac{f(x+h)}{f(x)}}, \ x\in R. \tag{17}$$

In this analysis the integral of the product is equal to a product of integrals and the derivative of a product is the product of the derivatives, i.e. not additive but the multiplicative property holds true.

7. Continuous Powerative Analysis.

If to rise one more step up then **in continuous case** we define the additive derivative

$$f^{\{I\}}(x) = \lim_{h\to 0}\sqrt[h]{\sqrt[f(x)]{f(x+h)}} \ , \tag{18}$$

the multiplicative derivative

$$\int_a^b d^{x} f(x) \quad = \lim_{\substack{\max|\Delta x_k| \to 0 \\ n\to\infty}} {}^{(\Delta x_n)} f\left(x_n\right)^{{}^{(\Delta x_{n-1})} f(x_{n-1})^{\cdot^{\cdot^{\cdot\, (\Delta x_0)} f(x_0)}}} \tag{19}$$

and the powerative derivative

$$f^{(I)}(x), \quad f^{[I]}(x), \; f^{\{I\}}(x)$$

and a discrete case we define the additive, multiplicative, powerative integrals as follows

$$\int_0^n f_k = \sum_{k=0}^{n-1} f_k \, ,$$

$$\int_0^n f_k = \prod_{k=0}^{n-1} f_k \, ,$$

$$\int_n^0 f_k = \mathop{\mathrm{P}}_{k=n-1}^{0} f_k \quad f_{n-1}^{f_{n-2}^{f_{n-3}^{\cdot^{\cdot^{\cdot^{f_2^{f_1^{f_0}}}}}}}}$$

respectively.

Remark 1. It is possible to define a new integral and derivative which contain three successive operations. An integral can be defined as a sum of products of powers and a derivative can be defined as a root of the ratio of differences.

Remark 2. Rising one step up we can define the other integral and derivative.

Remark 3. Such integrals and derivatives can be defined by means of four successive operations etc.

8. Discrete Additive Analysis.

Now we consider discrete analogues of the above operations. For this if in the definition of derivatives we take step *h*=1 then we'll have:

1) For additive derivatives in discrete case

$$f^{(1)}(x) = f(x+1) - f(x)$$

where $x \in N$ or $x \in Z$ that's why they write x instead of *n* , i.e.

$$f^{(1)}(n) = f(n+1) - f(n)$$

and this is written as

$$f^{(1)}{}_n = f_{n+1} - f_n \text{ , } n \in N \text{ or } n \in Z \text{ .} \tag{20}$$

If we recall the definition of algebraic progression from a secondary school , i.e.

$$a_{i+1} = a_i + d$$

starting from the second term then the first element can be given as follows

$$\begin{cases} a_{i+1} - a_i = d, \\ a_1 = \alpha. \end{cases}$$

From the definition (20) we have

$$\begin{cases} a_i^{(1)} = d, \\ a_1 = \alpha. \end{cases} \tag{21}$$

Definition of the common term of an algebraic progression brought us to Cauchy problem for a first order equation with discrete additive derivatives.

9. Discrete Multiplicative Analysis.

Now we discretize multiplicative derivative

$$f_n^{[1]} = \frac{f_{n+1}}{f_n} \text{ .} \tag{22}$$

Let us again return to a secondary school and remind of the definition of a geometric progression:

$$\begin{cases} a_{i+1} = a_i q \text{ ,} \\ a_1 = \alpha, \end{cases}$$

or

$$\begin{cases} \dfrac{a_{i+1}}{a_i} = q \text{ ,} \\ a_1 = \alpha, \end{cases}$$

or taking into account (22) we have

$$\begin{cases} a_i^{[1]} = q \text{ ,} \\ a_1 = \alpha. \end{cases} \tag{23}$$

The definition of a general term of a geometrical progression brought us to a Cauchy problem for a first order differential equation with discrete multiplicative derivatives.

10. Discrete Powerative Analysis.

A discrete powerative derivative is defined as follows:

$$f_n^{\{1\}} = \sqrt[f_n]{f_{n+1}} \, . \tag{24}$$

Thus, for the definition of these three derivatives no new operation is needed. Only seven algebraic operations are sufficient for them.

If there is given the equation

$$y_n^{[1]} = f_n, \; n \geq 0, \tag{25}$$

then we have a discrete powerative integral

$$y_n = f_{n-1}^{\,f_{n-2}^{\,f_{n-3}^{\,\cdots^{\,f_2^{\,f_1^{\,f_0}}}}}} \, . \tag{26}$$

Thus, we shall obtain that a discrete additive integral is

$$\int_0^n f_k = \sum_{k=0}^{n-1} f_k \, , \tag{27}$$

a discrete multiplicative integral is

$$\int_0^n f_k = \prod_{k=0}^{n-1} f_k \, , \tag{28}$$

a discrete powerative integral is

$$\int_n^0 f_k = \mathrm{P}_{k=n-1}^{0} f_k \quad f_{n-1}^{\,f_{n-2}^{\,f_{n-3}^{\,\cdots^{\,f_2^{\,f_1^{\,f_0}}}}}} \, . \tag{29}$$

All these discrete Cauchy problems and boundary value problems can be acquainted with in [1], [2-4], [5]-[10], [13], [14], [16]. Various problems for equations with continuous additive derivatives were investigated in [3], [17], [19]-[27].

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